The causal ladder and the strength of K-causality. I

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Abstract

A unifying framework for the study of causal relations is presented. The causal relations are regarded as subsets of $M \times M$ and the role of the corresponding antisymmetry conditions in the construction of the causal ladder is stressed. The causal hierarchy of spacetime is built from chronology up to K-causality and new characterizations of the distinction and strong causality properties are obtained. The closure of the causal future is not transitive, as a consequence its repeated composition leads to an infinite causal subladder between strong causality and K-causality - the A-causality subladder. A spacetime example is given which proves that K-causality differs from infinite A-causality.

1 Introduction

The causal relations are usually presented through their point based counterparts, namely the sets $I^{\pm}(x)$, $J^{\pm}(x)$, however the most natural and effective approach regards them as subsets of $M \times M$. It is convenient to define [12] the following sets on $M \times M$

$$I^+ = \{(p,q) : p \ll q\}, \quad J^+ = \{(p,q) : p \le q\}, \quad E^+ = \{(p,q) : p \to q\}.$$

Clearly, $E^+ = J^+ \backslash I^+$. Moreover, I^+ is open [13, Chap. 14, Lemma 3] [12, Prop. 2.16], $\bar{J}^+ = \bar{I}^+$, Int $J^+ = I^+$ and $\dot{J}^+ = \dot{I}^+$ [12, Prop. 2.17]. Once these sets are defined the conditions of chronology or causality are obtained as antisymmetry conditions on the corresponding relations on $M \times M$.

The approach with sets on $M \times M$ is also useful for the definition of new causal relations. For instance, about ten years ago Sorkin and Woolgar [17] defined the relations K^+ as the smallest closed subset $K^+ \subset M \times M$, which contains I^+, $I^+ \subset K^+$, and shares the transitivity property: $(x,y) \in K^+$ and $(y,z) \in K^+ \Rightarrow (x,z) \in K^+$ (the set of causal relations satisfying these properties is non-empty, consider for instance the trivial subset $M \times M$). This definition raised from the fact that J^+ while transitive is not necessarily closed whereas $\bar{J}^+$ while closed is not necessarily transitive.

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We shall see other examples in this work where the approach on $M \times M$ has proved not only useful but also superior to the one with the point based relations. For instance, the A-causality subladder introduced by Penrose will prove more natural than Carter's causal virtuosity subladder.

The aim of this work is to present a unifying framework for all the causal relations that have appeared in the literature. The various causality conditions are then traced back to conditions (usually to antisymmetry conditions) on these causal relations and the relationship between the different causality requirements becomes trivial and related to the inclusion of sets on $M \times M$.

Actually, since stable causality can be regarded as an antisymmetry condition on the Seifert future J^+_S, it could also be included in the present study. However, since there is an open issue as to whether stable causality coincides with K-causality I have preferred to leave these questions to a related work where the mentioned problem is studied in deep [11].

The work is organized as follows.

In section 2 a general approach to causal relations as subset of $M \times M$ initiated in [12] is introduced. The role played by the antisymmetry condition is stressed in view of its unifying role for the construction of the causal ladder. Not all the definitions or results presented in this section are later used. They are given because they hold whatever the causal relation considered and because the section is intended as a reference for future work, for instance for [11].

In section 3 new characterizations of the distinction and strong causality properties are obtained. In particular the past (resp. future) distinction is proved to follow from the antisymmetry of a causal relation termed D_p (resp. D_f). Strong causality is not characterized through the antisymmetry of a causal relation but, nevertheless, a similar useful result is obtained (theorem 3.3).

In section 4 the causal relations coming from the successive composition of $\bar{J}^+$ are considered. They give rise to a causal subladder which I clarify mentioning the different definitions that can be found in the literature. I shall mainly use the approach due to Penrose [15, Remark 4.19], who described the ladder explicitly, though I will not use the same terminology.

In section 5 I provide an example of spacetime which is infinite A-causal and yet non-K-causal. This result was long suspected but it had remained open so far. The information provided by the portion of the causal ladder displayed in figure 1 is then accurate and complete.

I refer the reader to [12] for most of the conventions used in this work. In particular, I denote with (M, g) a C^r spacetime (connected, time-oriented Lorentzian manifold), $r \in \{3, \dots, \infty\}$ of arbitrary dimension $n \geq 2$ and signature $(-, +, \dots, +)$. On $M \times M$ the usual product topology is defined.

The subset symbol $\subset$ is reflexive, $X \subset X$. With $J^+_U \subset U \times U$, I denote the causal relation on the spacetime U with the induced metric, so that $x \leq_U z$ reads $(x, z) \in J^+_U$. By *neighborhood* it is always meant an open set. The boundary of a set B is denoted $\dot{B}$.

K-causality
$\Downarrow$
A^{∞}–causality
$\Downarrow$
$\vdots$
$\Downarrow$
$A^{\frac{k}{2}}$–causality $\quad k > 2$
$\Downarrow$
$\vdots$
$\Downarrow$
A-causality
$\Downarrow$
Strong causality
$\Downarrow$
Distinction
$\Downarrow$
Causality
$\Downarrow$
Chronology

Figure 1: A portion of the causal ladder of spacetimes. An arrow between two properties $\mathcal{P}_1 \Rightarrow \mathcal{P}_2$ means that the former implies the latter and that there are examples of spacetimes in which the latter holds and the former does not hold.

2 Preliminaries

Due to the proliferation of causal relationships it is desirable to provide a common framework since many results can be derived in similar ways. The definitions given here are for most part compatible with those used in set theory [9]. A (binary) *relation* is a subset $R^+ \subset M \times M$, it is a *causal relation* if it contains I^+, $I^+ \subset R^+$, where I^+ is the chronological future for some given Lorentzian metric on M. It is *open* (resp. *closed*) if R^+ is open (resp. closed) as a subset of $M \times M$. Given R_1^+ and R_2^+ the composition $R_2^+ \circ R_1^+$ is the set

$$R_2^+ \circ R_1^+ = \{(x,z) \in M \times M : \exists y, (x,y) \in R_1^+ \text{ and } (y,z) \in R_2^+\}$$

The diagonal $\Delta = \{(x,x), x \in M\}$ is an identity for this composition that is, whatever R^+, $\Delta \circ R^+ = R^+ \circ \Delta = R^+$. The relation R^+ is *transitive* if $R^+ \circ R^+ \subset R^+$, that is for all $x,y,z \in M$, $(x,y) \in R^+$ and $(y,z) \in R^+ \Rightarrow (x,z) \in R^+$. It is idempotent if $R^+ \circ R^+ = R^+$ (for instance I^+ and J^+ are idempotent). It is *reflexive* if for all $x \in M$, $(x,x) \in R^+$, that is $\Delta \subset R^+$. It is *irreflexive* if for all $x \in M$, $(x,x) \notin R^+$, that is $R^+ \cap \Delta = \emptyset$. Clearly, every reflexive and transitive relation is idempotent. Note also that $\circ$ has a distributive property with respect to arbitrary unions of sets $\bigcup_\alpha A_\alpha^+ \circ \bigcup_\beta B_\beta^+ = \bigcup_{\alpha,\beta} (A_\alpha^+ \circ B_\beta^+)$. The properties of reflexivity and transitivity are also invariant under arbitrary intersections of relations sharing them.

A set B^+ is a left (resp. right) R^+-*ideal* if $B^+ \subset R^+$ and $R^+ \circ B^+ \subset B^+$

(resp. $B^+ \circ R^+ \subset B^+$), and an R^+-ideal if it is both a left and a right R^+-ideal (for instance I^+ is a J^+-ideal).

The relation R^+ is *antisymmetric* if for all $x, z \in M$,

$$(x, z) \in R^+ \text{ and } (z, x) \in R^+ \Rightarrow x = z$$

in which case M is said to be *R-causal* (I-causality coincides with chronology[1] and J-causality coincides with causality). *R-causality at $x \in M$* holds, if there is no point $z \in M$, $z \neq x$ such that $(x, z) \in R^+$ and $(z, x) \in R^+$. An important observation is that if $A^+ \subset B^+$ and B^+ is antisymmetric then A^+ is antisymmetric too. R^+ is *asymmetric* if $\forall x, z \in M$, $(x, z) \in R^+ \Rightarrow (z, x) \notin R^+$. Thus asymmetry is equivalent to antisymmetry and irreflexivity. Transitivity and irreflexivity imply asymmetry.

R^+ is a non-strict (or reflexive) *partial order* if it is reflexive, transitive and antisymmetric. It is a strict (or irreflexive) partial order if it is irreflexive and transitive. There is a one-to-one correspondence between non-strict and strict partial orders obtained by including or removing the diagonal. Indeed, the reader may easily prove that given the binary relation R^+, the reflexive relation $R^+ \cup \Delta$ is transitive and antisymmetric iff the irreflexive relation $R^+ \cap \Delta^C$ is transitive. Finally, strict partial orders can be characterized also as those relations which are asymmetric and transitive.

The reader may check that the next definition coincides with the usual one[10]

Definition 2.1. A triple (M, R^+, B^+), where M is a set, $R^+ \subset M \times M$ is a reflexive partial order and $B^+ \subset R^+$ is a irreflexive R^+-ideal is a *causal structure in the sense of Kronheimer and Penrose.*

Given R^+, the relation $R^- \subset M \times M$ is given by the set

$$R^- = \{(x, z) \in M \times M : (z, x) \in R^+\}, \tag{1}$$

in particular in the notation of the "R-causality" property the sign is omitted because the antisymmetric condition for R^+ coincides with that for R^-. Analogously, the diagonal Δ does not exhibit the plus sign because $\Delta^- = \Delta$.

Given R^+ the point based relations $R^+(x)$ and $R^-(x)$ are defined as

$$R^+(x) = \{y \in M : (x, y) \in R^+\}, \tag{2}$$

$$R^-(x) = \{y \in M : (y, x) \in R^+\}. \tag{3}$$

Note that $y \in R^+(x)$ iff $(x, y) \in R^+$ iff $x \in R^-(y)$.

R^+ is *partially open* (resp. *partially closed*) if for all $x \in M$ the sets $R^+(x)$ and $R^-(x)$ are open (resp. closed). Note that provided $I^+ \subset R^+$, if R^+ is

[1] Indeed, if chronology holds then the antisymmetry condition holds because the hypothesis "$(x, z) \in I^+$ and $(z, x) \in I^+$" is false for every $x, z \in M$. Conversely, if the antisymmetry condition holds then no closed timelike curve can exist otherwise one could find $x \neq z$, such that the hypothesis of the antisymmetry condition holds.

partially closed then it is reflexive, because $x \in \overline{I^+(x)} \subset R^+(x)$. It is trivial to prove that if R^+ is open (resp. closed) then it is partially open (resp. partially closed). Remarkably, the converse also holds provided $I^+ \subset R^+$ and R^+ is transitive as then next result shows.

Theorem 2.2. *Let (M,g) be a spacetime and let I^+ be the chronological relation. A transitive causal ($I^+ \subset R^+$) relation is open (resp. closed) iff it is partially open (resp. partially closed). Moreover, in this case it is also idempotent.*

Proof. Assume R^+ partially open (resp. partially closed) the other direction being trivial.
Open set case. Assume that R^+ is partially open and let $(x,z) \in R^+$ so that $x \in R^-(z)$. First let me show that R^+ is idempotent. Indeed, since it is partially open there is $y \in I^+(x) \cap R^-(z)$, and since $I^+ \subset R^+$, it is $I^+(x) \subset R^+(x)$, and finally $(x,y) \in R^+$ and $(y,z) \in R^+$, which proves $R^+ \circ R^+ = R^+$. Now, consider again arbitrary $(x,z) \in R^+$. Since R^+ is partially open there is a open neighborhood $U \ni x$ of compact closure such that $\bar{U} \subset R^-(z)$. Since R^+ is idempotent, $\bar{U}$ is covered by the open sets $\{R^-(y), \quad y \in R^-(z)\}$. Thus there is a finite number of events $y_i \in R^-(z)$, $i = 1, \dots, n$, and a subcovering of $\bar{U}$, $\{R^-(y_1), \dots, R^-(y_n)\}$. Define the open set $V = \bigcap_i R^+(y_i)$ so that $z \in V$, then for every $\bar{x} \in U$ and $\bar{z} \in V$, $(\bar{x}, \bar{z}) \in R^+$.

Closed set case. As already mentioned partial closure together with $I^+ \subset R^+$ implies reflexivity which implies the idempotent property. Let $(x,z) \in \bar{R}^+$ and let $(x_n, z_n) \in R^+$, $(x_n, z_n) \to (x,z)$. Take $s \in I^-(x)$ so that $x \in I^+(s)$. For large n, $x_n \in I^+(s)$ and hence $x_n \in R^+(s)$. By transitivity $z_n \in R^+(s)$ and by partial closure $z \in R^+(s)$, thus $s \in R^-(z)$ and taking the limit $s \to x$, again by partial closure, $(x,z) \in R^+$. □

With $R^+(x,y)$ it is denoted the set $R^+(x) \cap R^-(y)$. A set $V \subset M$ is R-*convex* if the causal relation R^+ is such that for every $x, z \in V$, $R^+(x,y) \subset V$ (J-convexity is the usual causal convexity). A spacetime is *strongly R-causal at* $x \in M$ if x admits arbitrarily small R-convex neighborhoods, that is, for every open set $U \ni x$ there is a R-convex neighborhood $V \subset U$, $x \in V$. The spacetime is *strongly R-causal* if it is strongly R-causal at every point (strong J-causality coincides with strong causality).

R^+ is *injective* if the maps on the set of parts of M, $R^\pm : M \to P(M)$, are injective, that is, $R^+(x) = R^+(z) \Rightarrow x = z$, and analogously for R^-.

Theorem 2.3. *The generic relation R^+ satisfies*

(a) If $R^+ : M \to P(M)$ or $R^- : M \to P(M)$ are injective and R^+ is transitive then R^+ is antisymmetric.

(b) If R^+ is antisymmetric and reflexive then both maps $R^+ : M \to P(M)$ and $R^- : M \to P(M)$ are injective.

(c) If R^+ is transitive and reflexive then the injectivity of the map $R^+ : M \to P(M)$ is equivalent to the injectivity of the map $R^- : M \to P(M)$. Moreover, the injectivity is equivalent to the antisymmetry.

Proof. Proof of (a). Assume $x \to R^+(x)$ is injective and that R^+ is transitive. Take $x, z \in M$ such that $(x, z) \in R^+$ and $(z, x) \in R^+$. Let $y \in R^+(x)$, since R^+ is transitive $y \in R^+(z)$, thus $R^+(x) \subset R^+(z)$. The other inclusion is analogous thus $R^+(x) = R^+(z)$ and by injectivity $x = z$.

Proof of (b). Assume R^+ is antisymmetric and reflexive and take $x, z \in M$ such that $R^+(x) = R^+(z)$. Then, because of reflexivity $x \in R^+(x) = R^+(z)$ and analogously $z \in R^+(x)$, thus by antisymmetry $x = z$.

Proof of (c). It is a trivial consequence of (a) and (b).

□

This theorem shows that under the assumption of transitivity and reflexivity the injectivity is equivalent to the antisymmetry and hence R-causality can be expressed in terms of the injectivity of the point based maps $R^+(x)$. That most of the causality conditions can be restated as an injectivity condition on some suitable causality maps from M to $P(M)$ has been checked in detail by I. Rácz [16]. The same happens for K-causality because, since K^+ is closed (and hence reflexive) and transitive by definition, it follows from theorem 2.3 that

Corollary 2.4. *The spacetime (M, g) is K-causal iff the map $x \to K^+(x)$ (or $x \to K^-(x)$) is injective.*

Theorem 2.2 implies

Corollary 2.5. *K^+ is the smallest transitive relation containing I^+ such that for every $x \in M$, $K^+(x)$ and $K^-(x)$ are closed.*

Proof. It follows from the fact that K^+ is the intersection of all the relations which are transitive, contain I^+ and are closed which by theorem 2.2 is the intersection of all the relations which are transitive, contain I^+ and are partially closed. □

3 Distinction and strong causality

In this section new characterizations of the distinction and strong causality properties are obtained.

A spacetime is future (resp. past) *distinguishing* if $I^+(x) = I^+(z)$ (resp. $I^-(x) = I^-(z)$) $\Rightarrow x = z$. For other characterizations not considered here see [12, Lemma 3.10]).

Theorem 3.1. *The spacetime (M, g) is future (resp. past) distinguishing if and only if for every $x, z \in M$, $(x, z) \in J^+$ and $x \in \overline{J^+(z)}$ imply $x = z$ (resp. $(x, z) \in J^+$ and $z \in \overline{J^-(x)}$ imply $x = z$).*

Proof. (Future case, the past case being analogous). If there is $x \neq z$ such that $(x,z) \in J^+$ and $x \in \overline{J^+(z)}$ then because of $(x,z) \in J^+$, $I^+(z) \subset I^+(x)$ while because of $x \in \overline{J^+(z)}$, $I^+(x) \subset I^+(z)$, thus $I^+(x) = I^+(z)$, that is (M,g) is not future distinguishing.

Conversely, if (M,g) is not future distinguishing there is $x' \neq z$ such that $I^+(x') = I^+(z)$. Since $z \in \overline{I^+(z)} = \overline{J^+(x')}$, let σ_n be a sequence of causal curves of endpoints x' and z_n, $z_n \to z$, and let σ^z be a limit curve of the sequence passing through z. Take $x \in \sigma^z \backslash \{z\}$, then $(x,z) \in J^+$ and $x \in \overline{J^+(x')} = \overline{J^+(z)}$, but $x \neq z$.

□

It is convenient to introduce the following subsets of $M \times M$,

$$D_f^+ = \{(x,y) : y \in \overline{I^+(x)}\,\}, \tag{4}$$

$$D_p^+ = \{(x,y) : x \in \overline{I^-(y)}\,\}. \tag{5}$$

Clearly, $J^+ \subset D_f^+ (\text{or } D_p^+) \subset \bar{J}^+$.

Definition 3.2. A spacetime is future (resp. past) reflecting if $D_f^+ = \bar{J}^+$ (resp. $D_p^+ = \bar{J}^+$). Equivalently, the spacetime is future (resp. past) reflecting if $(x,z) \in \bar{J}^+ \Leftrightarrow z \in \bar{J}^+(x)$ (resp. $(x,z) \in \bar{J}^+ \Leftrightarrow x \in \overline{J^-(z)}$). A spacetime is reflecting if it is both past and future reflecting.

The equivalence with other more traditional definitions of reflectivity follows from [12, Prop. 3.45] and [8, Prop. 1.3]). Note the different meanings of the terms *reflecting* which refers to the spacetime, and *reflexive* which refers to the causal relations.

Theorem 3.3. *The causal relations D_f^+ and D_p^+ are reflexive and transitive. Moreover, D_f^+ (resp. D_p^+) is antisymmetric, and hence a partial order, iff the spacetime is future (resp. past) distinguishing.*

Proof. The reflexivity is trivial because $x \in \overline{I^{\pm}(x)}$. The transitivity has been proved in [4, Claim 1]. In the future case the proof goes as follows, if $y \in \overline{I^+(x)}$ and $z \in \overline{I^+(y)}$, taken $w \in I^+(z)$, $z \in I^-(w)$ and since I^+ is open $y \in I^-(w)$, and again since I^+ is open $x \in I^-(w)$, $w \in I^+(x)$. Since w can be chosen arbitrarily close to z, $z \in \overline{I^+(x)}$. The antisymmetry of D_f^+ is equivalent to the injectivity of the map $x \to \overline{I^+(x)}$ because of theorem 2.3 point (c). Finally, since $\overline{I^+(x)} = \overline{I^+(y)} \Leftrightarrow I^+(x) = I^+(y)$, the said injectivity is equivalent the future distinction of the spacetime. □

Stated in another way, future distinction is equivalent to D_f-causality and past distinction is equivalent to D_p-causality. Note that this result is in one direction weaker than theorem 3.1 while in the other it is stronger. It implies that if there is a pair of distinct events such that $z \in \overline{I^+(x)}$ and $x \in \overline{I^+(z)}$ then there is another such that $(x',z') \in J^+$ and $x' \in \overline{I^+(z')}$.

A spacetime (M,g) is *strongly causal* at x if it admits arbitrarily small causally convex neighborhoods at x. It is strongly causal if it is strongly causal at every event.

Theorem 3.4. *The spacetime (M,g) is strongly causal if and only if for every $x,z \in M$, $(x,z) \in J^+$ and $(z,x) \in \bar{J}^+$ imply $x = z$.*

In particular, if $(x,z) \in J^+$, $(z,x) \in \bar{J}^+$ and $x \neq z$ then at the events belonging to $J^+(x) \cap J^-(z)$ the spacetime is non-strongly causal.

Proof. Assume that $x \neq z$ but $(x,z) \in J^+$ and $(z,x) \in \bar{J}^+$, and let us prove that (M,g) is not strongly causal at $r \in J^+(x) \cap J^-(z)$. Let $U \ni r$ be an arbitrary small neighborhood whose closure does not contain both x and z. Take $y \in I^+(r) \cap U$ and $w \in I^-(r) \cap U$, then $z \in I^+(w)$ and $x \in I^-(y)$. If σ_n is a sequence of causal curves of endpoints converging respectively to z and x then for sufficiently large n the first endpoint stays in $I^+(w)$ while the second endpoint stays in $I^-(y)$. As a result $y \in I^+(w)$ and a timelike curve connecting y to w can be chosen that passes arbitrary close to z and x and hence is not entirely contained in U. Thus the spacetime is not strongly causal at r.

Conversely, if (M,g) is not strongly causal then the characterizing property (ii) of [12, Lemma 3.22] does not hold, that is, there is $x \in M$, a neighborhood $U \ni x$ and a sequence of causal curves σ_n, not entirely contained in U, of endpoints x_n, z_n, with $x_n \to x$, $z_n \to x$. Let $C \ni x$ be a convex neighborhood whose compact closure is contained in another convex neighborhood $V \subset U$. Let $c_n \in \dot{C}$ be the first point at which σ_n escapes C. Since $\dot{C}$ is compact there is $c \in \dot{C}$, and a subsequence such that $c_k \to c$ and since V is convex, the causal relation on $V \times V$, J^+_V, is closed and hence $(x,c) \in J^+_V$ thus $(x,c) \in J^+$. But since $(c_k, z_k) \in J^+$ it is $(c,x) \in \bar{J}^+$ and yet $c \neq x$. □

A related result is [15, Theor. 4.31]. A trivial consequence of theorems 3.1 and 3.4 is

Corollary 3.5. *If (M,g) is strongly causal then it is distinguishing.*

A consequence of $\bar{J}^+ \subset K^+$ and theorem 3.4 is

Corollary 3.6. *If (M,g) is K-causal then it is strongly causal.*

Actually, a stronger result holds (theorem 4.1).

Theorem 3.7. *If a spacetime (M,g) is future reflecting (resp. past reflecting) then $\bar{J}^+ = K^+ = D^+_f$ (resp. $\bar{J}^+ = K^+ = D^+_p$). Moreover, if it is also future distinguishing (resp. past distinguishing) then it is K-causal and thus distinguishing.*

Proof. I give the proof in the future case. Future reflectivity reads $D^+_f = \bar{J}^+$ thus $\bar{J}^+$ is not only closed but also transitive, and it is the smallest relations with these properties containing I^+ hence $\bar{J}^+ = K^+ = D^+_f$. If the spacetime is also future distinguishing then $D^+_f = K^+$ is antisymmetric, i.e. the spacetime is K-causal. □

4 The A-causality subladder

The set $\bar{J}^+ \subset M \times M$ defines a causal relation which, following Woodhouse [18], it is also denoted A^+. Consistently with Woodhouse's notations and in agreement with the general definitions of section 2, I define the *almost causal* future and past of an event $x \in M$ as follows

$$A^+(x) = \{y \in M : (x,y) \in \bar{J}^+\}, \tag{6}$$

$$A^-(x) = \{y \in M : (y,x) \in \bar{J}^+\}. \tag{7}$$

These sets are clearly closed on M. According to Woodhouse[2] a spacetime is A-causal (or W-causal) if the causal relation $A^+ = \bar{J}^+$ on $M \times M$ is antisymmetric, that is if $(x,y) \in \bar{J}^+$ and $(y,x) \in \bar{J}^+$ imply $x = y$. Clearly, since $J^+ \subset \bar{J}^+ \subset K^+$ (recall theorem 3.4)

Theorem 4.1. *If (M,g) is K-causal then it is A-causal. Moreover, if (M,g) is A-causal then it is strongly causal.*

That the converse of these statements does not hold is shown by a classical example due to Carter [15, Fig. 25].

The *common* future $\uparrow S$ and past $\downarrow S$ of a set $S \subset M$ are open sets defined as follows

$$\uparrow S = \mathrm{Int}\{\bigcap_{x \in S} I^+(x)\} = \mathrm{Int}\{z \in M : \forall s \in S, s \ll z\}, \tag{8}$$

$$\downarrow S = \mathrm{Int}\{\bigcap_{x \in S} I^-(x)\} = \mathrm{Int}\{z \in M : \forall s \in S, z \ll s\}. \tag{9}$$

Note that $I^+(x) \subset \uparrow I^-(x)$ and $I^- \subset \downarrow I^+(x)$. It is not difficult to prove [1, Prop. 3] that $A^+(x) = \overline{\uparrow I^-(x)}$ and $A^-(x) = \overline{\downarrow I^+(x)}$, and hence, since $\uparrow I^-(x)$ and $\downarrow I^+(x)$ are open by definition, $\uparrow I^-(x) = \mathrm{Int}\, A^+(x)$ and $\downarrow I^+(x) = \mathrm{Int}\, A^-(x)$. Actually, a stronger result holds

Lemma 4.2. *It holds $\uparrow I^-(x) = Int\, D^+_p(x)$, $\overline{D^+_p(x)} = A^+(x)$, and $\downarrow I^+(x) = Int\, D^-_f(x)$, $\overline{D^-_f(x)} = A^-(x)$.*

Proof. It suffices to prove the characterization $D^+_p = \{(x,z) : \forall s \in I^-(x), s \ll z\}$ which implies $D^+_p(x) = \{z \in M : \forall s \in I^-(x), s \ll z\}$ and hence $\uparrow I^-(x) = \mathrm{Int}\, D^+_p(x)$. Indeed, if $(x,z) \in D^+_p$ then $x \in \overline{I^-(z)}$ thus taken $s \in I^-(x)$,

[2] As a matter of fact Woodhouse did not use this terminology. Note that the terminology for $A^+ = \bar{J}^+$ and $A^{\pm}(x)$, which are called *almost causal* futures, suggests to call the property of A-causality as *almost causality*. This terminology would follow by analogy with the causal and chronology conditions which can be expressed as antisymmetric conditions on the chronological and causal futures. Unfortunately, this terminology would suggest that A-causality is a weaker condition than causality while it is actually stronger. Thus I keep only the term *A-causality* which is also consistent with the general definitions at the beginning of section 2. Note that J.C. Park [14] called *almost causal* a spacetime which satisfies the relation $\bar{J}^+ = J^+_S$. This terminology is not used here.

$x \in I^+(s)$ and since I^+ is open $s \ll z$, thus $D_p^+ \subset \{(x,z) : \forall s \in I^-(x), s \ll z\}$. Conversely, if $(x,z) \in \{(x,z) : \forall s \in I^-(x), s \ll z\}$ then taken $s \in \overline{I^-(x)}$ it is $s \in I^-(z)$ and since s can be chosen arbitrarily close to x, $x \in \overline{I^-(z)}$, i.e. $\{(x,z) : \forall s \in I^-(x), s \ll z\} \subset D_p^+$. The other statements are proved analogously. □

Although $\bar{J}^+$ is not necessarily transitive the following result holds

Theorem 4.3. *The causal relations $B_p^+ = \{(x,y) : y \in \uparrow I^-(x)\}$ and $B_f^+ = \{(x,y) : x \in \downarrow I^+(y)\}$ are transitive, that is:*

(a) If $y \in \uparrow I^-(x)$ and $z \in \uparrow I^-(y)$ then $z \in \uparrow I^-(x)$.

(b) If $x \in \downarrow I^+(y)$ and $y \in \downarrow I^+(z)$ then $x \in \downarrow I^+(z)$.

Moreover, $\overline{B_p^-(y)} = \overline{I^-(y)} = D_p^-(y)$ and $\overline{B_f^+(x)} = \overline{I^+(x)} = D_f^+(x)$.

Proof. Case $\uparrow I^-$, the other case being analogous. Let $y \in \uparrow I^-(x)$ and $z \in \uparrow I^-(y)$. Since $\uparrow I^-(x) = \mathrm{Int} D_p^+(x) = \mathrm{Int}\{w \in M : x \in \overline{I^-(w)}\}$ there is a neighborhood $W \ni y$ such that for all $w \in W$, $x \in \overline{I^-(w)}$. Analogously there is a neighborhood $W' \ni z$ such that for every $w' \in W'$, $y \in \overline{I^-(w')}$. Choose $w \in W \cap I^-(y)$. Whatever the event $w' \in W'$, we have the chain $x \in \overline{I^-(w)}$, $(w,y) \in I^+$, $y \in \overline{I^-(w')}$. Since I^+ is open $x \in \overline{I^-(w')}$, and since $w' \in W'$ is arbitrary, and W' is an open neighborhood of z, $z \in \uparrow I^-(x)$.

Let us come to the proof of $\overline{B_p^-(y)} = \overline{I^-(y)}$. Let $x \in \overline{B_p^-(y)}$, there are x_n, such that $x_n \to x$ and $y \in \uparrow I^-(x_n) \subset D_p^+(x_n)$. Thus $x_n \in \overline{I^-(y)}$ and hence $x \in \overline{I^-(y)}$. Conversely, if $x \in \overline{I^-(y)}$ there are x_n, such that $x_n \to x$ and $x_n \in I^-(y)$ or equivalently $y \in I^+(x_n) \subset \uparrow I^-(x_n)$ which reads $(x_n, y) \in B_p^+$ and finally $x \in \overline{B_p^-(y)}$. The proof of $\overline{B_f^+(x)} = \overline{I^+(x)}$ is analogous.

□

Theorem 4.4. *If the spacetime (M,g) is past distinguishing then B_p^+ is antisymmetric. Analogously, if the spacetime (M,g) is future distinguishing then B_f^+ is antisymmetric.*

Proof. It is a consequence of the inclusions $\uparrow I^-(x) \subset D_p^+(x)$, and $\downarrow I^+(x) \subset D_f^-(x)$, given by lemma 4.2. Indeed, for instance, $(x,z) \in B_p^+$ and $(z,x) \in B_p^+$ reads $z \in \uparrow I^-(x)$ and $x \in \uparrow I^-(z)$, hence $z \in D_p^+(x)$ and $x \in D_p^+(z)$, which reads $(x,z) \in D_p^+$ and $(z,x) \in D_p^+$, and using the antisymmetry of D_p^+, $x = z$.
□

The point based relation $\uparrow I^-(x)$ (or $\downarrow I^+(x)$) is also nicely related to strong causality. Indeed, I. Rácz has shown [16, Prop. 3.1] that the map $\uparrow I^- : M \to P(M)$ (or $\downarrow I^+ : M \to P(M)$) is injective iff the spacetime is strongly causal.

Note that since B_p^+ is transitive $\uparrow I^-(x)$ is a future set. Analogously, since B_f^+ is transitive, $\downarrow I^+(x)$ is a past set. Thus using [2, Prop. 3.7]

$$A^+(x) = \overline{\uparrow I^-(x)} = \{y \in M : I^+(y) \subset \uparrow I^-(x)\}, \qquad (10)$$

$$A^-(x) = \overline{\downarrow I^+(x)} = \{y \in M : I^-(y) \subset \downarrow I^+(x)\}, \qquad (11)$$

which is the original, rather involved, definition of the sets $A^{\pm}(x)$ given by Woodhouse [18]. Akolia et al. [1, Prop. 10] noted, as also proved here, that the causal relation so defined coincides with $\bar{J}^+$ which explains why I started directly from this simpler definition of A^+.

After the introduction of the properties of chronology and causality by Kronheimer and Penrose [10] and strong causality and stable causality by S. Hawking [6], B. Carter [3] introduced the causal virtuosity hierarchy with the aim of making some order in the different causality requirements that were appearing in the literature. He showed that between strong causality and stable causality a denumerable sequence of, each time more demanding, properties could be defined. Carter's definitions were quite involved because he used the point based causal relation $J^{\pm}(x)$ instead of the more versatile J^+.

He defined a sequence of causal relations, let me denote them $\leq^n$, in which $\leq^0=\leq$, and $\leq^n$ was obtained by taking suitable closures and compositions of the previous causal relations $\leq^k, k < n$ (for an account see [5]). According to Carter's definition a spacetime is *n-th degree causally virtuous* (sometimes referred to as *n-th order strongly causal*), $n \geq 0$, if $x \leq^i z$ and $z \leq^j x$ with $i + j = n$ implies $x = z$. In particular it is *infinitely causally virtuous* if it is n-th degree causally virtuous for every n, that is if, whatever $i, j \in \mathbb{N}$, $x \leq^i z$ and $z \leq^j x$ implies $x = z$. According to Carter, 0-th degree causally virtuous spacetimes are simply the causal spacetimes, first causally virtuous spacetimes are the distinguishing spacetimes and the second causally virtuous spacetimes are the strongly causal spacetimes. That n-th degree causal virtuosity is different from $n+1$-th degree causal virtuosity was shown in an example due to Carter and published in [15, Fig. 25].

It must be said that given the causal hierarchy there is essentially no proof that the hierarchy is complete and a statement of this kind would probably make no sense at all. It can always happen that some day a new interesting causal property could be discovered which fits nicely in the hierarchy and simplifies some old statements and proofs. Carter's causal ladder had the merit to clarify this point but, at least in the author's opinion, the new levels introduced by Carter failed to prove particularly useful for the development of causality theory. In this respect K-causality is conceptually simpler but very similar to the *infinitely causally virtuous* property (if they are equivalent a proof would probably be complicated by the involved definition of the latter). It conveys the same ideas in a simplified way, and I think an almost definitive causal ladder should accommodate it in place of the causal virtuosity (sub)ladder.

The analogy between infinite causal virtuosity and K-causality becomes even more stringent if one recalls that the set K^+ can be built starting from J^+ via a *transfinite* induction [17, Lemma 14] in which at each step new pairs of events in the closure or obtained through transitivity are added, in a way which clearly

resembles that used by Carter for the definition of his $\le^n$ relations, but with the advantage that here no point based causal relation is used.

It is easy to prove the following

Theorem 4.5. *K-causality implies infinite causal virtuosity.*

Proof. The starting point of Carter's inductive process, i.e. J^+, is contained in K^+, and the construction of the sets corresponding to $\le^k$, $k > 0$, is obtained through compositions and closures that, due to the transitivity and closure properties of K^+ necessarily remain included in K^+. Thus $x \le^n z \Rightarrow (x, z) \in K^+$ and hence K-causality implies infinite causal virtuosity. □

The short account given by Penrose [15, Remark 4.19] of Carter's causal ladder has introduced some terminological confusion. He recognized that the main point of Carter's analysis was the possibility of constructing an infinite causal ladder between strong causality and stable causality, however instead of working with Carter's involved definitions he considered a simplified causal ladder which actually did not coincide with Carter's as it had wider steps (for instance the distinguishing property was not included). Unfortunately, due to this account, sometimes Penrose's causal ladder is identified with that introduced by Carter (see, for instance, [16]) a fact which may arise some confusion.

Penrose's ideas anticipated those by Woodhouse. Essentially, he considered a generalization of the notion of A-causal spacetime to arbitrary chains. In order to keep the connection with the A-causality property and the causal relations on $M \times M$ it is convenient to introduce Penrose's ladder as follows

Definition 4.6. The set $A^{+n} \subset M \times M$, $n \ge 1$, is the set of pairs (x_1, x_{n+1}), which can be connected by a n-chain $(x_i, x_{i+1}) \in A^+$, $i = 1 \dots n$, and $A^{+0} = \Delta$. In particular $A^{+1} = A^+$. The set $A^{+\infty} = \cup_{i=0}^{+\infty} A^{+i}$, is the set of the pairs of events which are connected by a chain of A-causally related events.

A spacetime (M, g) is $A^{n/2}$-*causal*, $n \ge 2$, if the existence of a cyclic n-chain $(x_i, x_{i+1}) \in A^+$, $i = 1 \dots n$, $x_1 = x_{n+1}$, implies $x_1 = x_i$, $i = 1, \dots, n$. A spacetime is A^∞-*causal* if it is $A^{k/2}$-causal for every integer $k \ge 2$ (or, equivalently, if $A^{+\infty}$ is antisymmetric).

This definition is motivated by the fact that if n is even then the $A^{n/2}$-causality property coincides[3] with the requirement of antisymmetry for the $A^{+n/2}$ causal relation in agreement with the general definitions of section 2. If instead n is odd there is no clear correspondence with a set on $M \times M$, and indeed no set $A^{+n/2} \subset M \times M$ has been defined for odd n. Clearly, since

[3]This statement can be proved as follows. Assume n even, and let the spacetime be $A^{n/2}$-causal according to definition 4.6, let $(x, z) \in A^{+n/2}$ and $(z, x) \in A^{+n/2}$ then there is a n-chain of A^+-related events connecting x to itself passing through z, thus $x = z$. Conversely, if the spacetime is $A^{n/2}$-causal according to section 2, then given a cyclic n-chain (x_i, x_{i+1}), $x_{n+1} = x_1$, which connects x_1 to itself then $(x_1, x_{n/2+1}) \in A^{+n/2}$ and $(x_{n/2+1}, x_{n+1}) \in A^{+n/2}$ which implies $x_{n/2+1} = x_1$. Now use the fact that A^+ is reflexive, and repeat the argument to obtain that $x_i = x_1$.

A^+ is reflexive $A^{(k+1)/2}$-causality implies $A^{k/2}$-causality, moreover they are distinct properties due to the usual example [15, Fig. 25]. Note also that $A^{+\infty}$ is transitive but not closed.

Since $A^+ \subset K^+$, it is $A^{+n} \subset K^+$ and hence

Theorem 4.7. *K-causality implies A^{∞}-causality.*

After the introduction of the infinitely causally virtuous property, S. Hawking [7] considered the possibility of its coincidence with the previously defined stably causal property. He expressed the opinion that this coincidence does not hold, without, as far as I know, providing an example of spacetime infinitely causally virtuous but non-stably causal. As I said any statement regarding Carter's causal properties is in general difficult to prove because of their involved definitions. It is then better to work with the $A^{k/2}$-causal ladder. In this respect it is meaningful to ask whether A^{∞}-causality coincides with K-causality. The spacetime example I provide in section 5 is A^{∞}-causal but is not K-causal.

5 A spacetime example

In this section I give an example of A^{∞}-causal non-K-causal spacetime (see figure 2).

Let (Λ, η) be 2+1 Minkowski spacetime and let (t, x^1, x^2) be canonical coordinates, $\eta = -\mathrm{d}t^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2$. The manifold M is obtained from Λ as follows. Remove the planes $t = 0$ and, $t = 1$. On the planes there are two (open in the plane topology) holes of the same size but with non-aligned centers which are not removed but rather identified (the correspondence between points is done respecting the Minkowskian parallel transport). The causal future of the lower hole is 'stopped' by a (closed) disk removed from Minkowski spacetime and of exactly the same size of the light cone at that height. The height of the disk from $t = 0$ is chosen so that the causal past of the upper hole reaches the edge of the removed disk at a point b (also removed). The lightlike geodesic segment ab in the boundary of the light cone issuing from the lower hole is also removed. The metric $g = \eta|_M$ is that induced from Minkowski spacetime.

A close inspection of this spacetime shows that if it were non-A^{∞}-causal then the two events $x, z \in M$ such that $(x, z) \in A^{+\infty}$, $(z, x) \in A^{+\infty}$ would necessarily stay in the lightlike inextendible geodesic γ obtained from the segment bc after removal of the endpoints. We may assume $z \in J^+(x)$. However, it is not possible that $x \in A^{+\infty}(z)$ indeed the set $A^-(x) = \{y : (y, x) \in \bar{J}^+\}$ is closed but has empty intersection with the closure of the causal future of the hole, thanks to the fact that the segment ab has been removed. Thus no matter how long is the chain of A-causally related events considered, none can connect z to x, thus $(z, x) \notin A^{+\infty}$ although $(z, x) \in K^+$ as the argument of the figure caption shows. Thus the spacetime is A^{∞}-causal and non-K-causal.

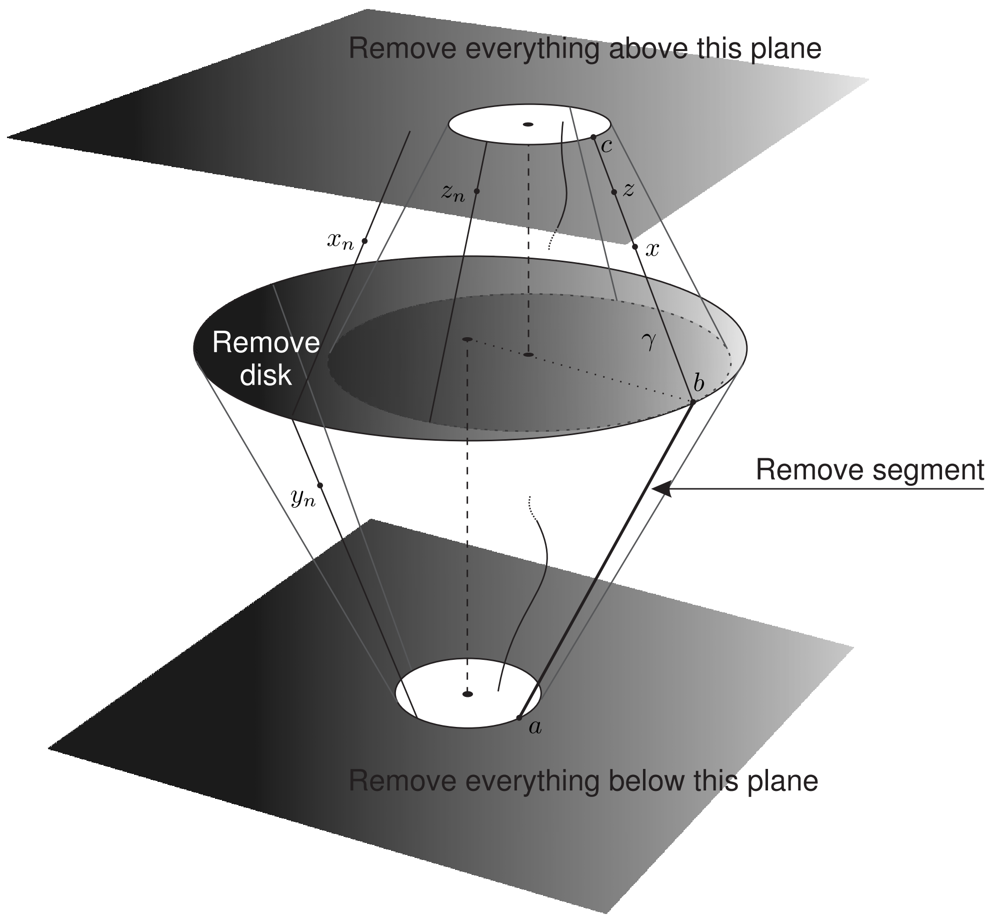

Figure 2: A non-K-causal but A^{∞}-causal spacetime is obtained from Minkowski spacetime by removing a disk, a geodesic segment, two planes with holes of the same size and by identifying the holes as shown in the figure. The metric is that induced from Minkowski spacetime. The events x_n, y_n and z_n are chosen so that $(z_n, y_n) \in \bar{J}^+$, thanks to the identification of the holes, and $(y_n, x_n) \in \bar{J}^+$. Thus $(z_n, x_n) \in K^+$ although $(x_n, z_n) \notin \bar{J}^+$. Since (z, x) can be regarded as a limit of a suitable sequence (z_n, x_n), it is $(z, x) \in K^+$ and also $(x, z) \in J^+$.

6 Conclusions

In this work a unifying approach to the study of causal relations has been presented in which the associated antisymmetry conditions play an important role. Indeed, through them the construction of the causal ladder becomes particularly clear as the relationship between the different causality requirements follows trivially from the inclusion of sets in $M \times M$.

Some new results on causality theory have been obtained. Among them the equivalence between openness (closure) and partial openness (partial closure) of transitive causal relations. The equivalence between antisymmetry and injectivity for reflexive and transitive relations. Some new characterization of K-causality and the K^+ relation, i.e. corollaries 2.4 and 2.5, or strong causality (theorem 3.4). The fact that the past and future distinction properties can be characterized through the antisymmetry conditions of suitable transitive and reflexive causal relations D_p^+ and D_f^+ (theorem 3.3).

Finally, other causal relations have been studied pointing out whether they are transitive or not, closed or open, partially closed or partially open. The A-causality subladder has been presented in detail and in the last section a spacetime example has been given which proves that K-causality differs from infinite A-causality.

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